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Acoustic Oscillations in Solid Propellant Rocket Chambers

By

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With 4 Figures

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Abstract

Acoustic Oscillations in Solid Propellant Rocket Chambers. Among the various kinds of periodic motions observed in rocket combustion chambers, the most common and simplest to analyze are those related to classical acoustic modes. If the amplitudes are small, the main perturbations of the familiar standing or travelling waves in a closed chamber are proportional to the Mach number of the mean flow. The correct equations describing the problem are here obtained from the general equations of motion by a limit process which will also provide equations for studying waves of finite amplitude. Subsequently, a single non-homogeneous wave equation is deduced, and solved by an iteration-perturbation procedure. The principal result is a simple formula for the complex frequency showing explicitly the effects of burning, suspended particles in the gases, the exhaust nozzle, and viscous wall forces as well as the mean flow itself. The last is particularly interesting since, owing primarily to the flow inward from the burning surface, the mean flow, if it is irrotational, never acts to damp modes which do not involve axial oscillations. As a particular application, the extensive data taken by BROWNLEE and MARBLE are interpreted to the extent that the linear analysis permits. A stability boundary was obtained from 250 firings of small cylindrical rockets, the principal variables being initial port diameter and length. The propellant did not contain metal particles, and it appears that the observations cannot be explained by the supposition that viscous damping associated with particles in the product gases was the main source of energy loss. Apparently dissipation at the head end, such as that associated with tangential wall shear forces, was an important loss. On the other hand, there is little doubt that if the combustion produces particles, the consequent dissipation is adequate to damp small amplitude waves.

Résumé

Oscillations acoustiques dans les moteurs-fusée à propergols solides. Parmi les divers mouvements périodiques observés dans les chambres de combustion, les plus fréquents et les plus simples à analyser sont ceux liés aux modes acoustiques classiques. Si les amplitudes sont faibles, les perturbations principales des ondes stationnaires ou progressives dans une chambre fermée sont proportionnelles au nombre de Mach de l'écoulement moyen. Les équations correctes décrivant le phénomène sont ici obtenues à partir des équations générales du mouvement par un processus de limite qui fournit aussi les équations nécessaires à l'étude des ondes d'amplitude finies. Ensuite, une équation d'onde unique et non-homogène en est déduite et résolue par un procédé d'itération — perturbation. Le résultat principal est une formule simple pour la fréquence complexe montrant explicitement les effects de la combustion des particules en suspension dans les gaz, de la tuyère, et des forces de viscosité à la paroi aussi bien que de l'écoulement moyen lui-même. Ce dernier est particulièrement intéressant puisque, principalement en raison de l'écoulement dirigé vers l'intérieur à partir de la surface de combustion, l'écoulement moyen, à condition d'être irrotationnel, n'amortit jamais les modes qui n'impliquent pas d'oscillations axiales. A titre d'application particulière, les données nombreuses obtenues par BROWNLEE et MARBLE sont interprétées dans les limites permises par une analyse linéaire. Une limite de stabilité a été obtenue à partir de 250 mises à feu de petites fusées cylindriques, les variables principales étant le diamètre initial de la chambre et la longueur. Le propergol ne contenait aucune particule métallique et il semble que les observations ne puissent être expliquées par l'hypothèse d'un amortissement visqueux associé aux particules dans les gaz brûlés comme source principale de perte d'énergie. Apparemment la dissipation du côté de la tête, telle que celle associée aux forces de viscosité tangentielles à la paroi, constitue une perte importante. D'autre part, il ne fait guère de doute que si la combustion produit des particules, la dissipation qui en résulte est suffisante pour amortir les ondes de faible amplitude.

Zusammenfassung

Schallschwingungen in Brennkammern von Feststoffraketen. Unter den verschiedenen Arten periodischer Schwankungen, die in Raketenbrennkammern beobachtet werden, sind die mit der klassischen, akustischen Art verwandten die häufigsten und die am einfachsten zu berechnenden. Sind die Amplituden klein, so sind die Hauptstörungen der bekannten stehenden oder wandernden Wellen in einer geschlossenen Kammer proportional der Mach-Zahl der mittleren Strömung. Die richtigen Gleichungen, die das Problem beschreiben, werden hier aus der allgemeinen Bewegungsgleichung durch einen Grenzübergang erhalten, der auch Gleichungen zum Studium von Wellen mit endlicher Amplitude ergibt. Danach wird eine einfache, nichthomogene Wellengleichung abgeleitet und durch eine Näherungs-Störungsrechnung gelöst. Das Hauptergebnis ist eine einfache Formel für die komplexe Frequenz, die deutlich die Auswirkungen des Abbrennens, der verteilten festen Teilchen, der Düse, der Wandkräfte auf Grund der Viskose und auch der mittleren Strömung selbst zeigt. Letzteres ist

besonders interessant, da — hauptsächlich infolge von Strömung nach innen von der Abbrandoberfläche — die mittlere Strömung, wenn es keine Rotationskomponente gibt, niemals jene Schwingungsarten dämpft, die keine Schwingungen in achsialer Richtung enthalten. Als besondere Anwendung werden die umfassenden Daten, die von BROWNLEE und MARBLE gewonnen wurden, bis zu dem Ausmaß interpretiert, das die lineare Analyse erlaubt. Eine Stabilitätsgrenze wurde mit 250 Versuchen mit kleinen zylindrischen Raketen erhalten, bei denen die hauptsächlichen Variablen der Anfangsseelendurchmesser und die Länge waren. Der Treibstoff enthielt keine Metallteile und es hatte den Anschein, daß die Beobachtungen nicht durch die Vermutung erklärt werden können, daß die Viskosedämpfung, welche mit Teilchen im Gas der Reaktionsprodukte verbunden ist, einen wesentlichen Anteil am Gesamtverlust darstellt. Scheinbar haben Verluste am vorderen Ende, wie zum Beispiel die mit den tangentialen Scherkräften an der Wand verbundenen, einen bedeutenden Anteil am Gesamtverlust. Andererseits besteht geringer Zweifel, daß, wenn die Verbrennung Teilchen erzeugt, der daraus folgende Verlust in der Lage ist, kleine Transversalwellen zu dämpfen.

Nomenclature

Some symbols which have standard meanings or which are defined in the text and used but briefly are not included in the following list.

$\bar{a}$ = average speed of sound
A_b = admittance function for a burning surface, $A_b = A_b^{(r)} + i\,A_b^{(i)}$
A_n = admittance function for a nozzle, $A_n = A_n^{(r)} + i\,A_n^{(i)}$
C_m = mass fraction of solid or liquid particles in the chamber gases
D_p = port diameter
e_0 = stagnation internal energy of the chamber gases
e_p = total energy of solid or liquid particles
E_α^2, E_N^2 = normalization constant [eq. (12)]
$\vec{F}$ = force exerted by particles on the gas
$\vec{\mathfrak{F}}$ = normalized form of $\vec{F}$
f = function defined by eq. (6)
$G(\vec{r}\mid\vec{r}_0)$ = Green's function defined by eqs. (7) and (8)
h = function defined by eq. (4)
k = normalized wave number, $k = \Omega + i\,\Lambda$
k_N = real eigenvalue for the normal modes, eq. (11), $k_N \equiv k_{lmn}$
k_l = axial part of k_N, $k_l = l\pi$
K_n = area ratio, $K_n = S_b/S_t$
m_1 = $m/\varkappa_{mn}$
$\bar{m}$ = average mass flow at the burning surface
m' = fluctuation in mass flow at the burning surface
M = Mach number, $M = u/\bar{a}$
M_b = mean flow Mach number at the burning surface
M_e = mean flow Mach number at the chamber exit
$\overline{M}$ = mean flow Mach number divided by M_b, $M_b\,\overline{M} = (\bar{u}/\bar{a})$
M' = Mach number of the fluctuations
M_c' = defined by eq. (17)
$\hat{n}$ = unit normal vector
r = radial distance normalized with respect to the port radius
R = radius of port
R_s = radius of solid particles
S_b = area of burning surface
S_t = area of throat
$\vec{u}_p$ = velocity of particles
z = axial distance normalized with respect to the port radius
β = attenuation constant for damping by particles
Γ = function of γ, $\Gamma = \left(\frac{\gamma - 1}{\gamma}\right)^{(\gamma+1)/(\gamma-1)}$
δ = parameter in eq. (25)
η = normalized variable, $\eta = p'/\gamma\,\bar{p}$
$\varkappa_{mn}$ = radial part of eigenvalues, eq. (11)
λ = total attenuation constant
Λ = imaginary part of the normalized wave number, $\Lambda = \lambda\,\omega/\bar{a}$
Λ_c = contribution to Λ associated with the mean flow (Table 2)
ν = kinematic viscosity
ϱ_p = average mass density of particles per volume of gas
σ = attenuation constant for damping by viscous shear stresses at the head end
τ_w, τ_p = decay constants
ψ, ψ_N = eigenfunctions
ω = natural frequency
Ω = normalized frequency, $\Omega = \omega\,R/\bar{a}$
$(\bar{\ })$ = average values
$(\ )'$ = fluctuations
$(\ )_s$ = evaluated at the surface
$(\ )_e$ = evaluated at the chamber exit

Introduction

Several kinds of unsteady behavior have been observed in solid propellant rockets, but perhaps the simplest to describe and analyze are oscillations with frequencies approximately equal to those found for classical acoustic modes. In particular, for a cylindrical burning surface, standing or travelling waves with radial and azimuthal motions have, until recently, been most troublesome. Much of the data for this kind of "combustion instability" tends to be incomplete, but the systematic experimental results reported by BROWNLEE and MARBLE [1] contain a great deal of information about steady waves. Although only those measurements will be interpreted in the following discussion, the analysis may be easily applied to other cases.

The rockets used in those experiments were cylindrical, with a five-inch inside diameter of the case, variable grain diameter, and variable grain length, from seventeen to forty-four inches; most of the firings involved thirty-one inch grains. An important feature of the work is that the composition of the propellant was carefully controlled and unchanged. Thus, apart from changes in grain temperature, which will not be treated here, the principal variables are geometric. A sketch of the rocket used and a representative time history of the mean chamber pressure during an unstable firing are reproduced in Fig. 1; the dashed line indicates the value of the pressure if there were no oscillations. Since changes in the mean pressure always accompanied oscillations, and were more easily measured than the fluctuations, the mean pressure itself was adopted as the primary dependent variable.

The instrumentation could detect a change of roughly one pound per square inch. Following ignition, before such changes could be observed, there was always an interval of 0.3—1.1 seconds, evidently due

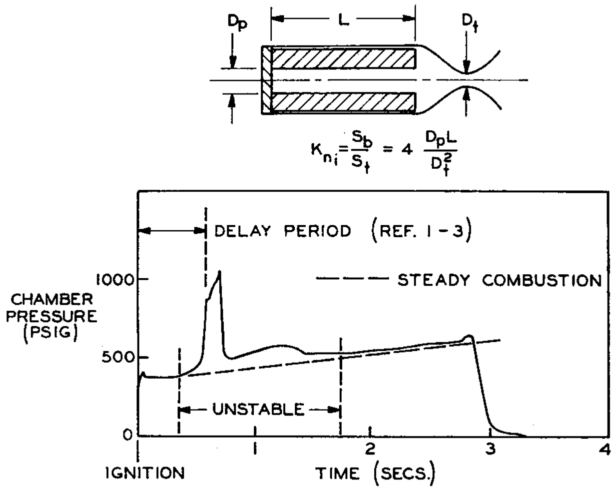

Fig. 1. Typical time history of mean chamber pressure (BROWNLEE and MARBLE [1])

to the time required for growth of the fluctuations in pressure. Subsequently, both the amplitude of the waves and the chamber pressure grow to large values. The linear analysis developed here will be restricted

required), and in fact, the same mode has been found, usually, by other experimenters. The stability boundary is approximately a straight line for much of the range of K_n and, coincidentally, falls on the locus of steady operation when $L = 31.0$ inches. For steady burning, the operating point moves away from the origin along a line $K_n = (\pi L/S_t)\, D_p$. Ideally, if the instantaneous values of K_n, D_p place the operating point above the stability boundary, then small oscillations are stable. (In the unstable region, away from the stability boundary, there are some cumulative effects of the history of the firing, which influence [2] the instantaneous values of $\Delta p/p$, but these will not be covered here.) A most important conclusion of the measurements is that the stability boundary definitely shifts upward for increasing length. This indicates that there is a net decrease of damping in the system. The interesting implications of the straight boundary and the change with length are easily deduced from the analysis.

There are also some data for dependence of the delay period on changes in geometry; the trends are not well established, but it will be seen that they appear to be consistent with the other observations. It should perhaps be emphasized that the delay period discussed in [1—3] is the time from ignition to the break in the curve of Fig. 1, whereas the interval referred to here is measured from ignition to the time when a 1 p. s. i. increase of mean pressure is first observed.

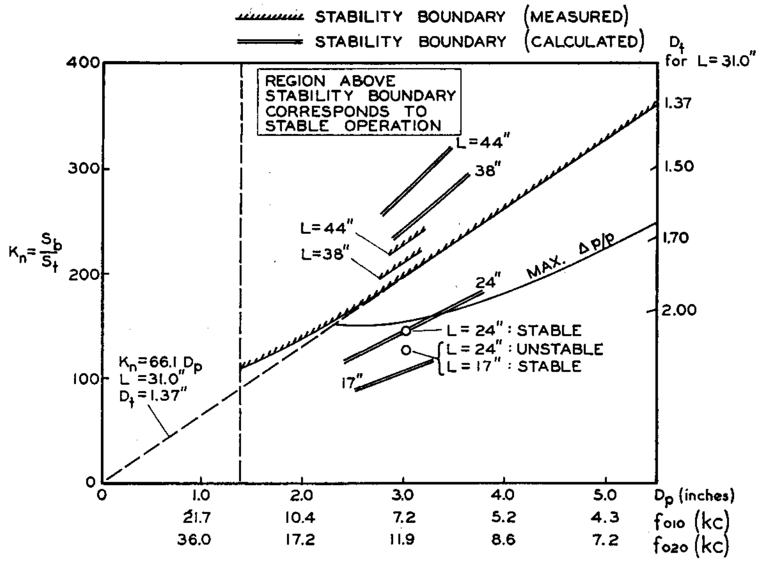

Fig. 2. Stability boundary, 010 Mode (BROWNLEE and MARBLE [1])

to the initial portion of this behavior, the incipient instability of small amplitude waves. Thus, only the stability boundary will be studied, and particularly the connections between the experimental observations and the various sources of damping and driving.

A summary of the main results to be discussed here is shown in Fig. 2. Data from 250 firings are involved; a thorough description may be found in BROWNLEE's dissertation [2]. Invariably, the same mode was excited (with no external disturbances

Thus, there are four gross aspects to be explained: the existence of the same mode in every unstable firing; the shape of the stability boundary; the shift of the boundary with changes of length; and effects of changes in geometry on the delay time. The experiments in fact contain much more, especially related to oscillations of finite amplitude and to the influence of different initial grain temperatures. Both lie outside the range of what follows.

It is apparent from measurements of the frequency, and its relation to the port diameter as indicated in

Fig. 2, that at least the small amplitude oscillations are closely allied to classical acoustic modes in a rigid chamber. On the other hand, the burning at the boundary, the consequent mean motion of the gases, and the presence of the sonic exhaust nozzle are obviously important differences. Indeed, the primary cause of the oscillations is the dependence of energy release on the states of both the product gases and the solid phase. All the features mentioned must be included in an analysis of the problem. Especially, it will appear that the mean flow fulfills an unexpected and important function.

Fortunately, work is greatly simplified by several crucial and valid assumptions. The linear burning rate is always small compared to the wave propagation speed: for the propellant used by BROWNLEE and MARBLE, it was never greater than one-half inch per second in steady combustion, whereas the speed of sound was around 3000 feet per second. Hence the rate of recession of the surface can be ignored in a treatment of steady waves. The combustion occurs within a very thin region, of the order of 10^{-3} to 10^{-1} inches thick. Accordingly, the coupling between the oscillations and the burning may be safely incorporated as a boundary condition placed on the waves. Finally, there are two small parameters. Near the stability boundary, the amplitude (ε) of oscillations is small, but also the mean flow Mach number is small. For example, in the experiments, the Mach number at the burning surface is $M_b \approx 0.005$ and at the chamber exit $M_e = (4\,L/D_p)\,M_b \approx 0.2-0.3$. When ε and M_b are zero, one has the usual acoustics problem. The proper set of differential equations containing all terms of first order in the mean motion is obtained from the equations for compressible flow by applying the limit process ε, $M_b \to 0$, but $\varepsilon/M_b \to 0$. In a similar way, the corresponding equations containing terms of higher order may be deduced.

From the coupled set of first order equations which result, one can form a second order equation for the pressure. Now the stability of standing, or travelling, waves is most conveniently studied by examining either the kinetic or potential energy. The latter is proportional to the square of the pressure, and since waves of pressure are observed, it is reasonable to use that quantity as the primary dependent variable. Thus the eq. (3) which is finally obtained for fluctuations of pressure may be regarded as an equation for the potential energy. It is an inhomogeneous wave equation, but the right hand side, as well as the associated inhomogeneous boundary conditions, is of order M_b, which therefore serves as the small parameter in an perturbation expansion.

In general, a Green's function may be introduced to convert the differential equation to an integral equation which is readily solved by a conventional iteration. The important result is a formula for the complex eigenvalues of the complete problem, exhibiting all perturbations to the appropriate order in ε and M_b. For the first approximation treated here, i.e., ε, $M_b \to 0$, a simpler calculation yields the formula directly. Other effects, such as dissipation due to solid particles in the gases and viscous stresses, may, and will, be included. Discussion of the data is based on that formula.

There exists, of course, a large amount of published work dealing with this problem, both general and, in one case, applied to BROWNLEE's results in particular. However, it appears that certain aspects have either not been interpreted correctly or else were obscured. One purpose of this work is to clarify several points and to incorporate the various parts of the problem within one analysis, free of unnecessary complications. Moreover, it will be shown that BROWNLEE's data yield more information than has previously been obtained. It is not the intention here to compile an exhaustive list of earlier works; some will be cited as required. Two recent useful surveys are by PRICE [4], an excellent summary of experimental results, and by HART and MCCLURE [5], a review of analytical work differing considerably from that presented here.

The discussion is split rather naturally into two main parts, the first dealing with the general analysis, followed by a detailed examination of BROWNLEE's experimental results.

Governing Equations and Boundary Conditions

This treatment of steady small amplitude waves in a solid propellant rocket parallels closely discussions [6, 7] of the corresponding problem in gas and liquid rocket chambers. In [7], the one-dimensional case is covered in detail, but here it is necessary to account for motions in three dimensions. Although the possibilities for conversion of liquid to gas and for significant energy release within the chamber need not be included, an additional volume force $\vec{F}$ will be carried in the momentum equation. Eventually $\vec{F}$ will be interpreted as the force due to viscous interaction between the gases and suspended solid particles. Other viscous effects and heat conduction will be ignored for the present; the equations for the fluctuations must therefore be deduced from the following set:

Conservation of Mass

$$\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\,\vec{u}) = 0$$

Conservation of Momentum

$$\frac{\partial\,(\varrho\,\vec{u})}{\partial t} + \nabla\cdot(\varrho\,\vec{u}\,\vec{u}) + \nabla p = \vec{F}$$

Conservation of Energy

$$\frac{\partial}{\partial t}(\varrho\,e_0 + \varrho_p\,e_p) + \nabla\cdot(\varrho\,\vec{u}\,e_0 + \varrho_p\,\vec{u}_p\,e_p) + \\ + \nabla\cdot(p\,\vec{u}) = 0$$

The mass average properties of the particles are ϱ_p, $\vec{u}_p$, e_p, the last being kinetic energy only since heat conduction between the gas and solid phases is not included. The momentum equation for the particles is

$$\frac{\partial}{\partial t}(\varrho_p\,\vec{u}_p) + \nabla\cdot(\varrho_p\,\vec{u}_p\,\vec{u}_p) = -\vec{F}.$$

By subtracting the scalar products of $\vec{u}$ with the momentum equation and $\vec{u}_p$ with the momentum equation for the particles, one may write the energy equation in the form

$$\frac{\partial}{\partial t}(\varrho\, C_v\, T) + \nabla\cdot(\varrho\, \vec{u}\, C_v\, T) + p\,\nabla\cdot\vec{u} = (\vec{u}_p - \vec{u})\cdot\vec{F}.$$

In all that follows, the equation of state for a perfect gas will be assumed so that the last equation may be combined with the continuity equation to give an equation for the pressure:

$$\frac{\partial p}{\partial t} + \gamma\, p\,\nabla\cdot\vec{u} = -\vec{u}\cdot\nabla p + (\gamma-1)\,(\vec{u}_p - \vec{u})\cdot\vec{F}.$$

It is a consequence of the equations for steady flow that to second order in the mean flow Mach number, the mean thermodynamic properties $\bar{\varrho}$, $\bar{p}$, $\bar{T}$ may be assumed uniform. To first order, however, variations in the Mach number itself must of course be considered. In order that the limit process cited above may be applied, the variables are first written as sums of mean (in time) and fluctuating parts: $p = \bar{p} + p'$, etc. For convenience, the dimensionless variables $\varepsilon\,\eta = p'/\gamma\,\bar{p}$, $\varepsilon\,\vec{M}' = \vec{u}'/\bar{a}$, $M_b\,\vec{\overline{M}} = \vec{\bar{u}}/a$ are introduced; η, $\vec{M}'$, $\vec{\overline{M}}$ are then regarded as quantities of order one. The usual conditions in a rocket are such that, within the chamber at least, the particle and gas velocities do not differ greatly; hence, in the first instance, one may assume that $\vec{F}'$ and $\vec{\overline{F}}$, which are proportional to the difference between the particle and gas velocities, are roughly of the same order as $\vec{u}'$ and $\vec{\bar{u}}$. The normalized forms are $\varepsilon\,\vec{\mathfrak{F}}' = \vec{F}'\,R/\bar{\varrho}\,\bar{a}^2$ and $\vec{\overline{\mathfrak{F}}}$.

Following the above substitutions and division by ε, the momentum equation and the equation for pressure may be put in the form[1]

$$\begin{aligned}\frac{\partial\vec{M}'}{\partial t} + \nabla\eta = &-M_b\,[\vec{\overline{M}}\cdot\nabla\vec{M}' + \vec{M}'\cdot\nabla\vec{\overline{M}}] + \vec{\mathfrak{F}}' - \\ &-\varepsilon\left\{\frac{\varrho'}{\varepsilon\,\bar{\varrho}}\frac{\partial\vec{M}'}{\partial t} + \vec{M}'\cdot\nabla\vec{M}' + M_b\frac{\varrho'}{\varepsilon\,\bar{\varrho}}[\vec{\overline{M}}\cdot\nabla\vec{M}' + \right.\\ &\left. + \vec{M}'\cdot\nabla\vec{\overline{M}}]\right\} - \varepsilon^2\frac{\varrho'}{\varepsilon\,\bar{\varrho}}\vec{M}'\cdot\nabla\vec{M}',\end{aligned}$$

$$\begin{aligned}\frac{\partial\eta}{\partial t} + \nabla\cdot\vec{M}' = &-M_b\,\{\vec{\overline{M}}\cdot\nabla\eta - (\gamma-1)\,\cdot\\ &\cdot(\vec{M}'_p - \vec{M}')\cdot\vec{\overline{\mathfrak{F}}} - (\gamma-1)\,(\vec{\overline{M}}_p - \vec{\overline{M}})\cdot\vec{\mathfrak{F}}'\} - \\ &-\varepsilon\,\{\vec{M}'\cdot\nabla\eta + \gamma\,\eta\,\nabla\cdot\vec{M}' - (\gamma-1)\,(\vec{M}'_p - \vec{M}')\cdot\vec{\mathfrak{F}}'\}\end{aligned}$$

Now let $\varepsilon\to 0$, $M_b\to 0$, but assume terms $o\,(\varepsilon)$ vanish more rapidly, in this process, than terms $O\,(M_b)$, i.e., $\varepsilon/M_b\to 0$. Clearly, if terms of first order in M_b are retained, one is left with the two equations

$$\frac{\partial\vec{M}'}{\partial t} + \nabla\eta = -M_b\,[\vec{\overline{M}}\cdot\nabla\vec{M}' + \vec{M}'\cdot\nabla\vec{\overline{M}}] + \vec{\mathfrak{F}}' \quad (1)$$

$$\frac{\partial\eta}{\partial t} + \nabla\cdot\vec{M}' = -M_b\,\vec{\overline{M}}\cdot\nabla\eta \quad (2)$$

[1] Note that some terms of second order in Mach number are missing in these equations due to the assumption that $\bar{p}$, $\bar{T}$ are uniform.

All quantities are dimensionless, the length and time scales being R and $R/\bar{a}$.

Although it appears that $\vec{\mathfrak{F}}'$ in (1) is not a perturbation, and hence should be included on the left hand side, this is true only so far as the limit process based on ε and M_b is concerned. In fact, there is a third small parameter measuring the size of $\vec{F}'$, and hence $\vec{\overline{F}}$. It depends on the properties of the gases, the solid material, and on the particle size. Moreover, there is yet another small parameter, δ, characterizing the interaction of small particles and acoustic waves; the relation between δ and $\vec{\mathfrak{F}}'$ will be demonstrated later. For the present, however, the point is that as a result of the special forms for the interactions between the gas and the particles, not only is $\vec{\mathfrak{F}}'$ properly treated as a perturbation in (1) but also the terms involving $\vec{\overline{\mathfrak{F}}}$ and $\vec{\mathfrak{F}}'$ in (2) are of higher order than those retained. Extended discussions relevant to the preceding remarks may be found in [8—10].

The appropriate wave equation for η is formed by differentiating (2) with respect to time and inserting (1) for $\partial\vec{M}'/\partial t$; for steady waves[2] having dimensionless wave number $k = \omega\,R/\bar{a}$,

$$\nabla^2\eta + k^2\,\eta = M_b\,h \quad (3)$$

$$h = i\,k\,\vec{\overline{M}}\cdot\nabla\eta - \nabla\cdot[\vec{\overline{M}}'\cdot\nabla\vec{M}' + \vec{M}'\cdot\nabla\vec{\overline{M}}] + \frac{1}{M_b}\nabla\vec{\mathfrak{F}}' \quad (4)$$

Conditions on the normal component $\hat{n}\cdot\nabla\eta$ of the gradient at the burning solid surface and on the exit plane are set in accord with eq. (1)

$$\hat{n}\cdot\nabla\eta = -i\,k\,\vec{M}'\cdot\hat{n} - M_b\,\hat{n}\cdot[\vec{\overline{M}}\cdot\nabla\vec{M}' + \vec{M}'\cdot\nabla\vec{\overline{M}}].$$

The term associated with the solid particles contributes nothing at the surface. If the chamber is closed by rigid walls and there is no burning, the normal component of velocity vanishes on the entire boundary; thus $\hat{n}\cdot\nabla\eta = 0$ on the surface, as in the classical problem. It is necessary now to find the first order (in M_b) correction to that case. Both functions η and $\vec{M}'$ will eventually be expanded in power series of M_b, with $\hat{n}\cdot\vec{M}' = \hat{n}\cdot\nabla\eta = 0$ at the ends of the chamber, $z = 0$, L/R, and on the cylindrical surface $r = 1$. But to first order, in general, $\hat{n}\cdot\vec{M}' \neq 0$ on $z = L/R$ because the flow passes into the nozzle, and on $r = 1$ because of the burning. In each case, the corrections are conveniently expressed in terms of admittance functions:

$$\frac{\hat{n}\cdot\vec{M}'}{\eta} = M_e\,A_n \qquad (z = L/R)$$

$$\frac{\hat{n}\cdot\vec{M}'}{\eta} = -M_b\,A_b \qquad (r = 1)$$

[2] In the experiments of BROWNLEE and MARBLE, it appeared that the mode was probably a standing wave. The travelling wave form of the same mode has been observed in other experiments.

In this way, the influences of the nozzle and the combustion are treated as problems distinct from the problem of stability. It can be shown that, for small amplitude oscillations, as the mean flow Mach number tends to zero, $M_e A_n \sim O(M_b)$ and $M_b A_b \sim O(M_b)$. The admittance function for the nozzle has been treated, for example, in [11] and [12]. Calculations of A_b have so far not been very successful; references to both analyses and experimental results may be found in [4] and [5].

The complex quantities A_n and A_b are introduced as definitions and represent the physical conditions that, because velocity fluctuations are non-zero on the surfaces, work may be done on the waves in the chamber by the bounding media; or work may flow to the surroundings. Thus, the nozzle may act to dissipate energy and the burning surface may either dissipate or provide energy, so far as the waves in the chamber are concerned. It is essentially "$p\,d\,v$" work which is involved. The frequencies of the various standing wave modes are also slightly affected by these boundary terms.

Hence the correct boundary conditions are $\hat{n} \cdot \nabla \eta = 0$ on $z = 0$ and

$$\hat{n} \cdot \nabla \eta = -M_b f \tag{5}$$

with

$$f = \hat{n} \cdot [\vec{\overline{M}} \cdot \nabla \vec{M}' + \vec{M}' \cdot \nabla \vec{\overline{M}}] + i k \eta \quad \left\{ \begin{array}{ll} -A_b & (r = 1) \\ \dfrac{M_e}{M_b} A_n & (z = L/R) \end{array} \right. \tag{6}$$

The way in which the admittance functions have been introduced shows clearly that the nozzle contributes nothing if, within the chamber, there is no fluctuating motion in the axial direction. The influence of the nozzle will be remarked upon further in later discussion.

First Order Solution for Eigenvalues

The calculations for finding a formula for the eigenvalues, correct to first order consistent with eq. (1) and (2), have been used previously in [6] and [7]. An iteration scheme is suggested by the form of eq. (3) and the boundary condition (5). Green's function for this problem satisfies

$$(\nabla^2 + k^2)\, G(\vec{r} \mid \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \tag{7}$$

where $\delta(\vec{r} - \vec{r}_0)$ is the delta function representing a harmonic point source at the position $\vec{r}_0$; thus G represents the disturbance received at $\vec{r}$. The boundary conditions chosen for G are that the normal component of its gradient vanishes on the entire bounding surface:

$$\hat{n} \cdot \nabla G = 0 \qquad (\vec{r} = \vec{r}_s) \tag{8}$$

Multiply (3) by G, (7) by η, subtract the two equations, and integrate the result over the volume V of the chamber: $0 < r < 1$, $0 < z < L/R$, $0 < \Phi < < 2\pi$ in cylindrical coordinates:

$$\int_V [G(\vec{r} \mid \vec{r}_0)\, \nabla^2 \eta - \eta \nabla^2 G]\, dV = = M_b \int_V G(\vec{r} \mid \vec{r}_0)\, h\, dV - \int_V \delta(\vec{r} - \vec{r}_0)\, \eta\, dV.$$

The second integral on the right side gives simply $\eta(\vec{r}_0)$. Green's identity may be applied to the left side, and after the boundary conditions (5) and (8) have been used, one has the formal solution for η:

$$\eta(\vec{r}) = M_b \int_V G(\vec{r} \mid \vec{r}_0)\, h\, dV_0 + + M_b \int_S G(\vec{r} \mid \vec{r}_{0s})\, f\, dS_0. \tag{9}$$

The variables $\vec{r}$ and $\vec{r}_0$ have been interchanged and use of the symmetry property $G(\vec{r} \mid \vec{r}_0) = G(\vec{r}_0 \mid \vec{r})$ (see, for example, eq. (12) — the property can be proved more generally); $\vec{r}_{0s}$ denotes values of $\vec{r}_0$ on the surface S.

A useful expression for $G(\vec{r} \mid \vec{r}_0)$ is the expansion in normal modes of the unperturbed problem. Assume $G = \Sigma A_\alpha \psi_\alpha(\vec{r})$ where

$$(\nabla^2 + k^2)\, \psi_\alpha(\vec{r}) = 0 \tag{10}$$

with

$$\vec{n} \cdot \nabla \psi_\alpha = 0 \qquad (\vec{r} = \vec{r}_S)$$

and α stands for three indices l, m, n. The eigenfunctions ψ_α are, of course,

$$\psi_\alpha(\vec{r}) = \cos(k_l z) \cos(m \Phi)\, J_m(\varkappa_{mn} r)$$

where $m = 0, 1, 2, \ldots$ and the required surface conditions are met if

$$k_l = l \pi \frac{R}{L} \qquad (l = 0, 1, 2, \ldots).$$

$$\left[\frac{d J_m(\varkappa_{mn} r)}{dr}\right]_{r=1} = 0$$

The roots $\varkappa_{mn}$ of the last equation are well known and tabulated in Table 1; the eigenvalues (real, dimensionless wave numbers) are

$$k_\alpha^2 = k^2_{lmn} = k_l^2 + \varkappa^2_{mn}. \tag{11}$$

Table 1. *Values of the Roots* $\varkappa_{mn}$ *of* $d J_m(\varkappa)/d\varkappa = 0$

m \ n	0	1	2
0	0	3.83	7.02
1	1.84	5.33	8.53
2	3.05	6.71	9.96

Substitution of the expansion for G into eq. (7) and use of the orthogonality properties of the functions ψ_α yields the A_α, and G is:

$$G(r \mid r_0) = \sum_{\alpha} \frac{\psi_\alpha(\vec{r})\,\psi_\alpha(\vec{r}_0)}{E_\alpha^2\,(k^2 - k_\alpha^2)}. \tag{12}$$

The normalization constant is

$$E_\alpha^2 = \int \psi_\alpha^2\,dV = \int_0^{L/R} \cos^2(k_l z)\,dz \cdot$$

$$\cdot \int_0^{2\pi} \cos^2(m\,\Phi)\,d\Phi \int_0^1 J_m^2(\varkappa_{mn} r)\,r\,dr =$$

$$= \frac{L\,\pi}{2\,R}\left[1 + \frac{\sin l\,\pi}{l\,\pi}\right] \begin{cases} (J_0^2(\varkappa_{0n}) + J_1^2(\varkappa_{0n})) & m = 0 \\ \frac{1}{2}\left(1 - \frac{m^2}{\varkappa^2_{mn}}\right) J_m^2(\varkappa_{mn}) & m \neq 0 \end{cases}$$

In accord with earlier remarks concerning experimental observations, one expects the solution for η to have the form of an unperturbed normal mode and a correction $O(M_b)$:

$$\eta(\vec{r}) = \psi_{lmn}(\vec{r}) + O(M_b).$$

Such a result is indeed found after inserting (12) into (9) and separating one term, say $\alpha = N$, from the series:

$$\eta(\vec{r}) = \psi_N(\vec{r})\,\frac{M_b}{E_N^2\,(k^2 - k_N^2)} \cdot$$

$$\cdot \left[\int_V \psi_N(\vec{r}_0)\,h\,dV_0 + \int_S \psi_N(\vec{r}_{0s})\,f\,dS_0\right]$$

$$+ M_b \sum_{\alpha \neq N} \frac{\psi_\alpha(\vec{r})}{E_\alpha^2\,(k^2 - k_\alpha^2)} \cdot$$

$$\cdot \left[\int_V \psi_\alpha(\vec{r}_0)\,h\,dV_0 + \int_S \psi_\alpha(\vec{r}_{0s})\,f\,dS_0\right].$$

The required form is obtained provided the coefficient of ψ_N is unity; this condition gives a formula for the eigenvalue, also having the form of the unperturbed value plus terms $O(M_e)$:

$$k^2 = k_N^2 + \frac{M_b}{E_N^2}\left[\int_V \psi_N(\vec{r}_0)\,h\,dV_0 + \int_S \psi_N(\vec{r}_{0s})\,f\,dS_0\right] \tag{13[1]}$$

Since the terms in brackets are multiplied by M_b, the first order correction for k^2 is computed by setting $\eta = \psi_N$, the zeroth order mode shape, everywhere in h and f.

The formula for k^2 is more quickly deduced by considering the equations for η and ψ_N. Multiply (3) by ψ_N, (10) by η, subtract the equations, integrate over volume, and use Green's theorem to find

$$\int_S [\psi_N \nabla\eta - \eta\nabla\psi_N]\cdot\hat{n}\,dS_0 =$$

$$= M_b \int_V \psi_N\,h\,dV_0 - (k^2 - k_N^2)\int_V \psi_N\,\eta\,dV_0.$$

Thus, with the proper boundary conditions inserted in the surface integrals,

$$k^2 = k_N^2 + \frac{M_b}{\int \psi_N\,\eta\,dV_0} \cdot$$

$$\cdot \left[\int_V \psi_N(\vec{r}_0)\,h\,dV_0 + \int_S \psi_N(\vec{r}_{0s})\,f\,dS_0\right].$$

Setting $\eta \approx \psi_N$ everywhere gives k^2 to order M_b, a formula identical with (13). The advantage of using Green's function is that higher order terms can be computed systematically. Hence, if one wishes to investigate problems involving either effects of order M_e^2 (or higher) or cases of finite amplitude waves ($\varepsilon \neq 0$), the first step is to find the correct equations by use of an appropriate limit process. The Green's function introduced here may still be used, and the iteration procedure may be carried out further; the calculations are considerably more tedious.

Two equations are provided by the formula (13) since the eigenvalue $k = \Omega + i\,\Lambda$ is complex. For vanishingly small Λ, i.e., near the stability boundary where the dissipation or production of energy is small, the formulas for Ω^2 and Λ are:

$$\Omega^2 = k_N^2 + \frac{M_b}{E_N^2}\left[\int_V \psi_N(\vec{r}_0)\,h^{(r)}\,dV_0 + \right.$$

$$\left. + \int_S \psi_N(\vec{r}_{0s})\,f^{(r)}\,dS\right], \tag{14}$$

$$\Lambda = \frac{M_b}{2\,\Omega\,E_N^2}\left(\int_V \psi_N(\vec{r}_0)\,h^{(i)}\,dV + \right.$$

$$\left. + \int_S \psi_N(\vec{r}_{0s})\,f^{(i)}\,dS\right]. \tag{15}$$

It is useful and quite easy to interpret these results. First, a few preliminary remarks are necessary. Write the perturbed form of the momentum equation for steady waves as

$$\vec{M}' = \frac{i}{k}\nabla\eta + M_b\,\vec{M}'_c - \frac{i}{k}\vec{\mathfrak{F}}' \tag{16}$$

so that $\vec{M}'_c$ is defined to be

$$\vec{M}'_c = \frac{i}{k}\,[\vec{\vec{M}}\cdot\nabla\vec{M}' + \vec{M}'\cdot\nabla\vec{\vec{M}}]. \tag{17}$$

Thus, $\bar{\varrho}\,\bar{a}\,M_b\,\vec{M}'_c$ is the fluctuation of momentum in a unit volume because of two effects: convection of fluctuations by the average motion, and convection of mean momentum by velocity fluctuations. The total fluctuation is the sum of (17) and the contribution $(i/k)\nabla\eta$ due to the gradient of pressure fluctuations as in ordinary acoustics. To zeroth order in the mean flow Mach number, $\vec{M}' = (i/k)\nabla\eta$, so that the first order approximation to $\vec{M}'_c$ is

$$\vec{M}'_c \approx -\frac{1}{k^2}\,[\vec{\vec{M}}\cdot\nabla(\nabla\eta) + (\nabla\eta)\cdot\nabla\vec{\vec{M}}]. \tag{17a}$$

Hence, while the part $(i/k)\nabla\eta$ of $\vec{M}'$ is $\pi/2$ out of phase with η (all phases are measured with respect

[1] To first order, the same formula is obtained for the traveling modes, $\psi_N \sim \exp i\,(m\,\Phi - k_N\,t)$.

to pressure fluctuations), $\vec{M}'_c$ to this order lags η by π. Averaged over a period, $\eta\,(i/k)\,\nabla\eta$ is zero, but the average of $\eta\,\vec{M}'_c$ is non-zero and represents work done by the pressure fluctuations in the gas moving with velocity $\bar{a}\,\vec{M}'_c$.

Since changes in the mean density are negligible for small Mach number, the mean flow may be approximately represented by the solution for incompressible flow; in the present case of a cylindrical rocket with burning along the entire length,

$$\vec{\overline{M}} \approx 2\,z\,\hat{e}_z - r\,\hat{e}_r \tag{18}$$

where $\hat{e}_z$, $\hat{e}_r$ are unit vectors in the axial and radial directions, respectively. (Note that the actual Mach number is $M_b\,|\,\vec{\overline{M}}\,|$.)

In view of the above comments, h and f are

$$h \approx i\,k\,[\vec{\overline{M}}\cdot\nabla\eta + \nabla\cdot\vec{M}'_c] + \frac{1}{M_b}\nabla\vec{\mathfrak{F}}',$$

$$f \approx i\,k\,[-\vec{M}'_c\cdot\hat{n} + \vec{M}'_s\cdot\vec{\hat{n}}],$$

where $M'_s\cdot\dfrac{\hat{n}}{\eta} = A$, the appropriate admittance function for the nozzle or burning surface; $\bar{a}\,\vec{M}'_s\cdot\hat{n}$ is therefore the fluctuation in velocity normal to the non-rigid boundary. Because $\vec{M}'_c$ given by (17a) is real, all terms explicitly involving the mean motion contribute nothing to the real parts of h and f. Consequently, the frequency, Eq. (14), is but slightly changed from the classical values for a closed chamber, by the burning, the nozzle, and possibly by the effect of $\vec{\mathfrak{F}}'$.

Now $\psi_N \approx \eta$ in the formula for k^2 and $E_N{}^2 = \langle\eta^2\rangle$, the dimensionless volume integral of the square of the pressure, which is twice the potential energy (normalized) in the chamber. Since the energy of the waves, to zeroth order in M_b, is half potential and half kinetic, $\bar{\varrho}\,\bar{a}^2\,R^3\,E_N{}^2$ represents the total energy associated with the waves. The formula for Λ may be written

$$2\,\Lambda = \frac{M_b}{\langle\eta^2\rangle}\left\{\int\limits_V \vec{\overline{M}}\cdot\nabla\frac{\eta^2}{2}\,d\,V - \int\limits_S \eta\,\vec{M}'_c\cdot\hat{n}\,d\,S + \int\limits_V \eta\,\nabla\cdot\vec{M}'_c\,d\,V + \int\limits_S \eta\,\vec{M}'^{(r)}_s\cdot\hat{n}\,d\,S + \int\limits_V \eta\,\frac{\nabla\cdot\vec{\mathfrak{F}}'^{(i)}}{M_b\,k}\,d\,V\right\}. \tag{19}$$

The time average of the potential energy in steady waves with no dissipation is $\eta^2/4$, so that $\eta^2/2$ represents the time average of total energy. Since $\nabla\cdot\vec{\overline{M}} = 0$, the first integral in (19) can be written as $\int\nabla\cdot\vec{\overline{M}}\,(\eta^2/2)\,d\,V = \int(\eta^2/2)\,\vec{\overline{M}}\cdot\hat{n}\,d\,S$ which obviously represents the time-averaged convection of energy through the boundary by the mean flow.

The next surface integral represents work done at the boundary associated with the fluctuating velocity $\bar{a}\,\vec{M}'_c$ explained above. That it is modified by the following volume integral is associated with the removal of mechanical energy from the original energy equation when the equation for pressure was deduced earlier. It is a curious coincidence that the net effect of these contributions is exactly equal to the loss of energy due to convection by the mean flow. Let $\varphi = \vec{\overline{M}}\cdot\nabla\eta$; because $\nabla\cdot\vec{\overline{M}} = \nabla\times\vec{\overline{M}} = 0$, (17a) becomes $\vec{M}'_c = -\nabla\varphi/k^2$ and the volume integral in question may be transformed by use of Green's theorem to

$$-\frac{1}{k^2}\int\limits_V \eta\,\nabla^2\varphi\,\nabla^2 d\,V = -\frac{1}{k^2}\int\limits_V \varphi\,\nabla^2\eta\,d\,V - \frac{1}{k^2}\int\limits_S [\eta\,\nabla\varphi - \varphi\,\nabla\eta]\cdot\hat{n}\,d\,S.$$

Since η is approximated by ψ_N and k^2 by $k_N{}^2$ here, so that $\nabla^2\eta \approx -k^2\eta$ and $\nabla\eta\,n \approx 0$, one is left with

$$\int\limits_V \eta\,\varphi\,d\,V - \frac{1}{k^2}\int\limits_S \eta\,\nabla\varphi\cdot\hat{n}\,d\,S \equiv \int\limits_V \vec{\overline{M}}\cdot\frac{\nabla\eta^2}{2}\,d\,V + \int\limits_S \eta\,\vec{M}'_c\cdot\hat{n}\,d\,S.$$

The surface integral is cancelled by the second integral in (19) and the assertion is proved. Note that this conclusion is independent of the geometry and of the mean flow providing only that it is incompressible potential flow. The influence of the mean flow has been investigated elsewhere [13], in a different way, giving the same final result.

The remaining contributions to (19) have clear meanings; the next to last represents net work done by the fluctuation pressure moving with velocity $\bar{a}\,M'_s$, and the last represents the rate of dissipation of energy due, for example, to suspended particles. Hence, $2\,\Lambda$ properly represents the time average of energy lost from the waves in the chamber divided by the total energy $(\mathfrak{E}) : 2\,\Lambda = -(d\mathfrak{E}/dt)/\mathfrak{E}$. The meaning of Λ, but not the interpretation of its various pieces, follows directly from the fact that $\mathfrak{E} \sim \exp(-2\,\Lambda\,t)$. Within this linear analysis, it is therefore a simple matter to account for other sources of energy loss by adding to (19) terms having the form $(d\mathfrak{E}/dt)/\mathfrak{E}$.

This discussion also leads to neater forms for Ω^2 and $2\,\Lambda$:

$$\Omega^2 = k_N{}^2 + \frac{M_b}{E_N{}^2}\left\{-\int\limits_S \eta^2\,A^{(i)}\,d\,S + \int\limits_V \eta\,\frac{\nabla\cdot\vec{\mathfrak{F}}'^{(r)}}{M_b\,k_N}\,d\,V\right\} \tag{20}$$

$$2\,\Lambda = \frac{M_b}{E_N{}^2}\left\{2\int\limits_S \frac{\eta^2}{2}\,\vec{\overline{M}}\cdot\hat{n}\,d\,S + \int\limits_S \eta^2\,A^{(r)}\,d\,S + \int\limits_V \eta\,\frac{\nabla\cdot\vec{\mathfrak{F}}'^{(i)}}{M_b\,k_N}\,d\,V\right\}, \tag{21}$$

in which A is $-A_b$ on the burning surface and $(M_e/M_b)\,A_n$ on the nozzle entrance plane, providing the mode considered involves axial motion (i.e., $l \neq 0$).

Some General Aspects of the Formulas

It is by now evident that the numerical calculations required for a particular case are surprisingly simple. The formulas (20) and (21) are quite generally applicable, within the linear approximations, and one might have expected that the contributions associated with the mean flow would be complicated. But only the mean flow normal to the boundary must be known, and that is frequently established simply by considering conservation of mass. For a cylindrical burning surface, as in the rockets used by BROWNLEE, the flow field is represented by (18). One then finds that with $\eta \approx \psi_N$, all direct effects of the mean flow are contained in the simple formula for A_c, valid for all modes,

$$\frac{1}{2\,E_N^2}\int\limits_S \eta^2\,\vec{M}\cdot\hat{n}\,dS = A_c =$$

$$= \left(1 - \frac{\sin l\pi}{l\pi}\right) - \frac{m_1^2}{1 - m_1^2}$$

where $m_1 = m/\varkappa_{mn}$. Table 2 contains the values of A_c for the lower modes of practical concern. It is particularly interesting that for $l = 0$, the mean flow is never a source of damping (i.e., $A_c \leqslant 0$ always) and for fixed n, its effectiveness as a driving force increases with m; in fact, it becomes indefinitely large since $m/\varkappa_{mn} \to 1$ for $m \gg 1$. The reason that the average flow does not act to dissipate energy is that the rate of convection of energy (of the waves) into the chamber at the burning surface is greater than the loss at the exit plane. As shown in the previous section, an equally important contribution is the work done, on the waves, associated with the motion represented by $\vec{M}'_c$.

Table 2. *Values of A_c for the Seventeen Lowest Modes. Upper Values Are for* $l \neq 0$, *Lower Values for* $l = 0$

m \ n	0	1	2
0	1 —	1 0	1 0
1	0.682 − 0.418	0.963 − 0.0366	0.986 − 0.0140
2	0.232 − 0.768	0.902 − 0.0978	0.958 − 0.0422

Moreover, if the strength of a mode is measured by the maximum amplitude of the wave, then for a given strength, the total energy of the modes, proportional to $(1 - m_1^2)$, decreases as m and n increase. This is a consequence of the mode shape described by the Bessel function, the amplitude of which decreases as its argument increases: the part of the mode near the burning surface has smaller amplitude than the part near the axis. And as m, n increase, more relatively low amplitude waves fit into the space $0 < r < 1$, so that the energy is less. Purely axial modes ($l \neq 0$, $m = n = 0$), on the other hand, have energy independent of l because the amplitude is the same for all waves within any mode. Thus, a given amount of damping or driving (i.e., energy lost or gained per unit time) has a relatively greater effect on the higher modes because the energy change in unit time constitutes a greater fraction of the total energy in the chamber.

Similarly, the influence of the burning surface, expressed in the term

$$-\frac{M_b}{E_N^2}\left[\int\limits_S \eta^2\,A_b^{(r)}\,dS\right]_{r=1} = \frac{-\,2\,M_b\,A_b^{(r)}}{(1 - m_1^2)}$$

increases numerically with m if $A_b^{(r)}$ is fixed. If $A_b^{(r)} > 0$, the interaction between the acoustic wave and the burning surface acts to drive the waves, augmenting the action of the mean flow for $l = 0$. However, as the frequency increases, A_b becomes negative, and for sufficiently large m and n, the mean flow tends to drive a mode but the combustion tends to damp, contrary to the situation existing for l, m, n small.

The importance of accounting for the influence of the mean flow must be particularly emphasized. Since both are proportional to the mean flow Mach number, the effects of the mean flow and the burning surface are best considered together as

$$M_b\left[A_c + \frac{1}{E_N^2}\int \eta^2\,A_b^{(r)}\,dS\right]$$

The terms in the brackets are generally of comparable size. Clearly then, the common practice of assessing the stability characteristics of a propellant on the basis of the admittance function alone may be misleading. Further consequences of this combination and numerical results are discussed in the next section.

If l is non-zero, the damping offered by the nozzle appears in A as the term

$$\frac{M_b}{2\,E_N^2}\left[\int\limits_S \psi_N^2\,\frac{M_e}{M_b}\,A_n^{(r)}\,dS\right]_{z=L/R} = M_e\,\frac{R}{L}\,A_n^{(r)}$$

when the nozzle fairs smoothly into the chamber. (Otherwise, this result is multiplied by the ratio of nozzle entrance area to chamber cross-sectional area [7].) The dependence of $A_n^{(r)}$ on geometry, frequency, etc., is discussed elsewhere; the interesting point here is that if $A_n^{(r)}$, and hence the amount of energy dissipated in unit time, is fixed, then the effect of the nozzle varies inversely with L. If L is increased, for example, and the amplitude of the mode is unchanged, then the total energy in the chamber is increased proportionately; therefore, a fixed amount of energy lost is a smaller fraction of the energy in the chamber. Clearly, the influence of any source of damping at the ends of the chamber will be weakened in the same way when the chamber length is increased.

On the contrary, if the dissipation is distributed uniformly throughout the chamber, both the amount of energy lost per unit time and the total energy increase with L; the ratio $(d\mathfrak{E}/dt)/\mathfrak{E}$ varies with L only through possible dependence on frequency. Evidently then, a great deal can be learned from experiments involving changes of length.

Interpretation of Some Experimental Results

In BROWNLEE's experiments, the mode observed is defined by $m = 1$, $l = n = 0$ (0 1 0 mode) for which there are no fluctuations parallel to the axis. Hence, the nozzle has no influence on the waves in the chamber. The suggestion ([14], p. 468) that the shift of the stability with length of chamber, as shown in Fig. 2, due to the nozzle therefore cannot be correct. At first thought, it is perhaps puzzling that this is the case. For if one imagines a wave with radial fluctuations passing into the nozzle, the side walls and variable mean flow conditions must evidently cause both reflection and transmission of energy. The point is that there is, within the nozzle, a complicated system of reflected and impinging waves which interfere so as to produce zero axial velocity at the entrance plane. In this way, there can be no net work done at the chamber exit.

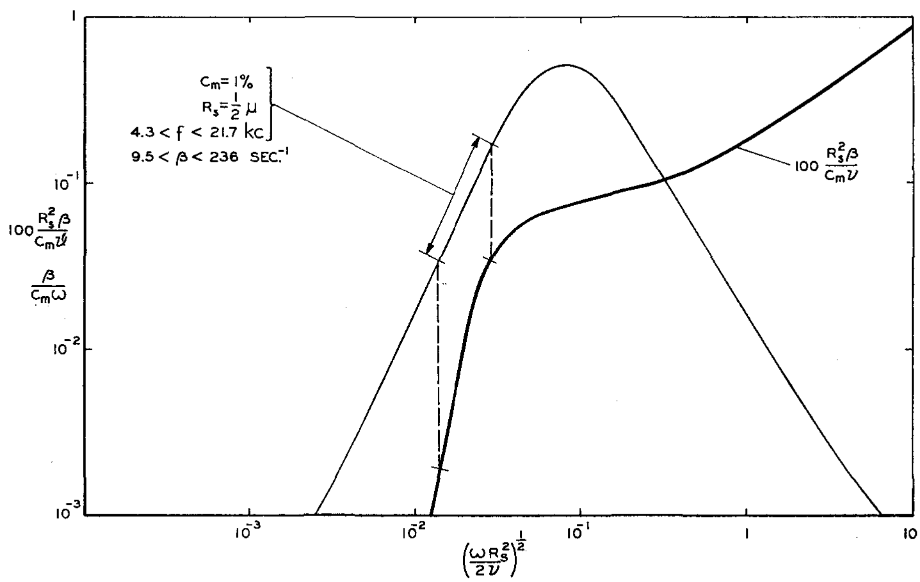

Fig. 3. Attenuation of sound by solid particles (EPSTEIN and CARHART [16])

There has been considerable interest (e.g. [4]) in the damping of waves by particles of condensed aluminum oxide formed after the burning of propellants containing pure aluminum. One is free to assume that the integral involving $\vec{\mathfrak{F}}'$ represents the average rate of dissipation of energy, per unit volume, due to the particles. And if one assumes further that the concentration of particles is uniform in the chamber, the calculations of EPSTEIN and CARHART [10] apply without change; the associated attenuation constant is β (sec^{-1}), which becomes $\beta\ R/\bar{a}$ in dimensionless form.

The argument of the preceding section shows that the dependence of the stability boundary on length cannot be contained in β. For while there is a marked variation of β with frequency (see Fig. 3 and eq. (25)), changes of length do not affect the frequency of the 0 1 0 mode. Another source of damping must be suggested, and it must be located either at the exit or at the head end. The possibility exists that uneven flow at the exit end of the grain is a cause, but a number of firings involving changes of the free volume between the end of the grain and the nozzle throat apparently had negligible effect on the stability characteristics. Consequently, it appears that whatever the cause, it is at the head end; the attenuation constant will be denoted by σ (sec^{-1}). After these details have been incorporated in (20) and (21), the formulas for $l = 0$ are

$$\Omega^2 = \varkappa_{mn}^2 - \frac{2\,M_b}{(1 - m_1^2)}\,A_b^{(i)}, \tag{22}$$

$$\Lambda = -\,M_b\,\frac{m_1^2}{(1 - m_1^2)} - \frac{M_b}{(1 - m_1^2)}\,A_b^{(r)} + (\beta + \sigma)\,\frac{R}{\bar{a}}. \tag{23}$$

Possible changes in frequency due to the particles and damping at the head end have been ignored.

The classical value for the frequency is but slightly changed by the burning, a correction probably too small to measure accurately in most cases. Since $A_b^{(i)}$ appears to be positive as defined here (a measurement has been reported in [15]), the frequency ought to be somewhat less than the value for a chamber with rigid walls. However, such a small change will usually be masked by uncertainties in the speed of sound and hence the true value of the frequency.

The equations for the stability boundary is given by (23) with $\Lambda = 0$. For comparison with Fig. 2, M_b is expressed in terms of K_n, $M_b = \Gamma/(\sqrt{\gamma}\,K_n)$ with $\Gamma = \left(\frac{\gamma - 1}{2}\right)^{(\gamma +)/(\gamma - 1)}$:

$$K_n = \frac{2\,\Gamma\,\bar{a}}{\sqrt{\gamma}\,D_p}\,\frac{(A_b^{(r)} + m_1^2)}{(\beta + \sigma)\,(1 - m_1^2)}. \tag{24}$$

When $L = 31''$, the experimental stability boundary is accurately represented as $K_n = 66.1\,D_p$. So far as the author is aware, the only other attempt to predict this result appears in [14], which did not include either effects of the mean flow or the dissipation represented by σ. Then for $\gamma = 1.25$, $A_b^{(r)} = 0.25$, and $\beta = 236/D_p^2$, one finds $K_n = 66.1\,D_p$. Although

$A_b^{(r)}$ is not independent of frequency, according to experiments, the value 0.25 is reasonable (see Fig. 4).

If the important damping is indeed due to solid particles, and the results of EPSTEIN and CARHART can be used, the constant β may be computed easily and is shown in Fig. 3. Measurements [16—18] have partially confirmed the computations. The curve for $\beta/C_m\,\omega$ shows that for a given frequency and mass fraction C_m of solids, the damping reaches a maximum as a function of particle size. The number density of particles, for C_m fixed, varies inversely with the cube

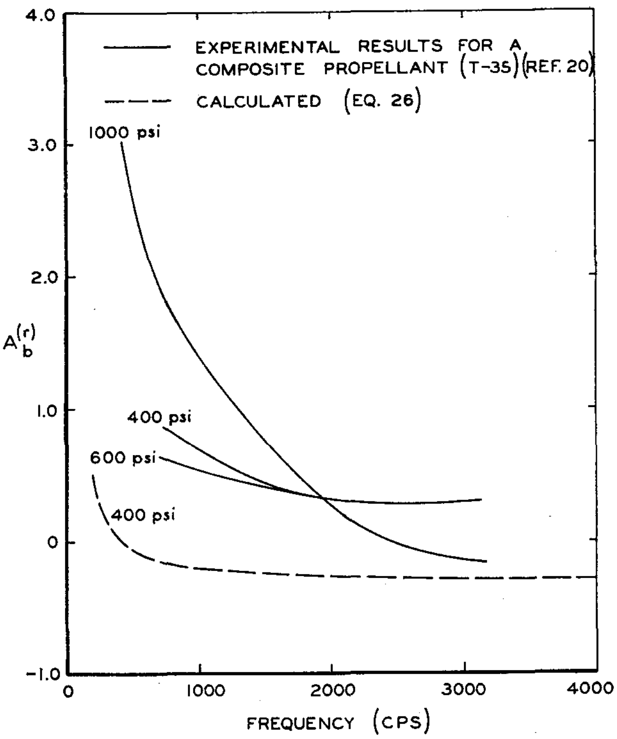

Fig. 4. Response of a burning propellant to steady pressure oscillations

of the particle radius R_s, while the drag force is always proportional to R_s^2. Thus, for large particles, the force per unit volume of gas decreases as R_s^{-1}. On the other hand, for $R_s \to 0$, it is a result of the analysis that the perturbation due to a particle is proportional to R_s^2. Thus, the force per unit volume varies as $R_s^2\,R_s^2/R_s^3 = R_s \to 0$. For small, medium, and large particles, the attenuation constant may be approximated by the formulas

$$\beta\,R/\bar a = \begin{cases} \dfrac{4}{9}\,\delta\left(\dfrac{\varrho_s}{\bar\varrho}\right)^2\left(\dfrac{R_s\,\omega}{2\,\nu}\right)^2 & (\omega\,R_s^2/2\,\nu \ll 9\,\bar\varrho/4\,\varrho_s) \\ \dfrac{9}{4}\,\delta & (9\,\bar\varrho/4\,\varrho_s \ll \omega\,R_s^2/2\,\nu \ll 1) \\ \dfrac{9}{4}\,\delta\left(\dfrac{R_s^2\,\omega}{2\,\nu}\right)^{\frac{1}{2}} & (1 \ll \omega\,R_s^2/2\,\nu) \end{cases} \qquad (25)$$

It is the parameter $\delta = (\bar\varrho/\varrho_s)\,(C_m\,\nu\,k_N/\omega\,R_s^2)$ which was earlier referred to in connection with the crude ordering of the terms involving $\vec{\mathfrak{F}}'$.

In [14], it was assumed that 0.5 micron particles were present, in a mass concentration of 1 per cent; for the conditions in the rocket, these constitute "small" particles, and with $\bar a = 3085$ ft/sec, $\nu = 5.73 \times 10^{-5}$ ft²/sec, $\beta = 236/D_p^2$ if D_p is expressed in inches. The range for β for the range of D_p in BROWNLEE's experiments is indicated in Fig. 3; it falls close to the optimum range.

Several criticisms may be directed at that treatment of the experiments. The proposal that particles might be the important source of damping was based [14] mainly on the possibility of small amounts of "smoke" in the exhaust. But such particles normally have sizes in the range 0.01—0.1 microns [23]. The propellant used by BROWNLEE contained no aluminum and therefore might be expected to yield at best particles in this size range. Later experiments [18, 19] using a similar propellant containing aluminum disclosed the presecne of particles of the size assumed in [14]. Moreover, as remarked previously, there is no possibility for explaining the obvious influence of changes of length if only damping by particles is significant. It should be emphasized that when the propellant does contain aluminum, there seems little doubt that the particles do provide substantial damping, often sufficient to prevent excitation of the acoustic modes.

Now since all fluctuating motions are parallel to the head end, it seems reasonable to examine the action of viscous shear stresses in that region. It is true that the motions are not likely to be simply laminar, if only because the cavity in the head contained instrumentation, but perhaps at least a lower bound on the damping may be fixed by considering simple oscillating flow parallel to a plate. This is a familiar problem in viscous flow theory, and for the present case, σ may eventually be expressed as

$$\sigma = \frac{\sqrt{\nu\,\omega}}{\sqrt{2}\,D_r\,E_n^2}\int_0^1 r\,dr\int_0^{2\pi}|\vec M'|^2\,d\Phi = \frac{\nu\,\omega}{2\,L}\,\frac{(\varkappa_{mn}/k_N)^2}{\left(1+\dfrac{\sin l\,\pi}{l\,\pi}\right)}.$$

It happens that the terms $O(M_b)$ in $\vec M'$ contribute nothing to σ. For $l = 0$, $\varkappa_{10} = 1.84$, and the values of ν and $\bar a$ quoted above, $\sigma = 21.4/L\,\sqrt{D_p}$. Both L and D_p are measured in inches. Thus, σ is considerably less than the value for β used above. For this reason, it was discarded in [14], although it was cited as a possible source of damping. If, instead, β is simply ignored (partly on the basis that there probably were not significant particles in the gases), eq. (24) is

$$K_n = 0.0935\,\frac{\Gamma\,\bar a}{\sqrt{\gamma}}\,\frac{L}{\sqrt{D_p}}\,(A_b^{(r)} + m_1^2). \qquad (26)$$

For the right hand side to be 66.1 D_p. $A_b^{(r)}$ must obviously depend on D_p, i.e., on frequency. The admittance function for the propellant used is not known, but it is trivial to compute what it must be to match the observed stability boundary. Fig. 4 shows the result and a few experimental measurements for comparison. The observed values for $A_b^{(r)}$ are inferred from direct measurement of pressure fluctuations; the errors involved are unknown. Even with the over-simplified representation, however, the computed results based on (26) seem reasonable. In fact, σ is probably underestimated, so that the calculated values of $A_b^{(r)}$ are too low. Only a qualitative comparison should be made since a different, though similar, propellant was used in the experiments, and the admittance function is very sensitive to changes of composition [22]. The values required here do show the tendency for a peak at lower frequencies; this is associated principally with the lag of heat transfer in the solid and is always observed.

Perhaps the most compelling reason for believing that there is significant dissipation of energy at the head end is that eq. (26) contains roughly the correct dependence on length, for the reason given in the preceding section. The stability boundary for $L = 31''$ has been matched exactly, giving $K_n = 2.13\, L D_p$, and boundaries for other values of L calculated. These are shown as the double lines in Fig. 2. Even quantitatively, the agreement with observations is fairly good; it is difficult to assess credibly the errors associated with the measurements, but since the damping available is less for the larger chambers, there is likely to be more error in the measurements for $L > 31''$.

The reason that the 0 1 0 mode was observed seems to be contained in the balance between the effects of the mean flow (Λ_c) and the behavior of the propellant response function (A_b) with frequency. The modes involving axial vibrations are heavily damped by the mean flow since Λ_c is sufficiently positive that $\Lambda_c - A_b^{(r)}) > 0$. But if $l = 0$, $\Lambda_c - A_b^{(r)} < 0$, and as m increases, Λ_c becomes increasingly negative, so that the question arises: why isn't a mode for larger m observed? Some cases have been reported [20], but those are rare, apparently because the propellants commonly used cannot support pressure oscillations having the higher frequencies.

Typically, the maximum in $A_b^{(r)}$ falls below roughly one kilocycle, as illustrated in Fig. 4. The asymptotic behavior is deduced from the relation

$$A_b = \gamma \frac{m'/\overline{m}}{p'/\overline{p}} - 1$$

where m' is the fluctuation in mass flux into the chamber at the burning surface. For high frequencies, the combustion process responds less to oscillations of pressure and $m' \to 0$; $A_b^{(r)} \to -1$, and presumably $\Lambda_c - A_b^{(r)} > 0$. Work must then be done by the waves at the surface. Thus, one reaches the surprising conclusion that for the higher modes, the driving by the mean flow is insufficient to overcome the damping offered by the burning: those modes cannot be sustained.

The sizes of rockets are often such that "higher modes" in this connection means $m > 1$. In large rockets, the frequency of the 0 1 0 mode may be only a few hundred cycles per second, even less than the frequency at which $A_b^{(r)}$ reaches its maximum. It is then not possible to make an intelligent prediction of the mode to be expected without having further knowledge of $A_b^{(r)}$. Note that for $\omega \to 0$, but still large enough so that the oscillations are approximately isentropic in the chamber, $A_b \to \gamma\, n - 1$, where n is the exponent in the linear burning rate law p^n; $\Lambda_c - A_b^{(r)}$ may be positive or negative. But even if $\Lambda_c - A_b^{(r)} < 0$, the dissipation of energy by particles may easily be large enough to damp all modes.

For the case of 0.5 micron particles considered above, $\beta = 2.36 \times 10_1\, C_m/D_p^2$ and if $D_p = 3$ inches, $\beta = 26.2\ \text{sec}^{-1}$ when $C_m = .01$. The corresponding value for σ in a 31″ rocket is .398 sec^{-1}. Propellants containing large amounts of aluminum may produce mass fractions C_m of 30 per cent and higher; one therefore expects $\beta/\sigma \gg 1$ even if σ has been greatly underestimated. Indeed, the particle damping may be large enough to dominate the factors $\Lambda_c - A_b^{(r)}$ in agreement with experience, showing that the difficulty of unstable acoustic modes is generally overcome by adding aluminum to the propellant.

Finally, there is fragmentary evidence that the growth rate of the oscillations observed by Brownlee and Marble is closer to what one would expect if the damping occurred at the head end than if dissipation by particles were responsible. Suppose that the delay time for the change of mean pressure noted in Fig. 1 corresponds to an exponential growth rate of time constant $\tau_p : \Delta\,\overline{p}/\overline{p} \sim 1 - e^{t/\tau}\ p$. The smallest observable increments in pressure were 1 p.s.i. out of 400 p.s.i., so that $t/\tau_p \approx 1/400$, and for a delay time of one-quarter second, $1/\tau_p \sim .01\ \text{sec}^{-1}$. Now, within the linear analysis, the growth rate for waves is

$$\frac{1}{\tau_w} \sim -\lambda \frac{\overline{a}}{R} = \frac{M_b\, \overline{a}}{(1 - m_1^2)\, R} (A_b^{(r)} + m_1^2) - (\beta + \sigma).$$

When conditions are not far removed from those defining the stability boundary, the difference between the terms on the right hand side may be expected to be, say, 1/10 of the magnitude of the separate terms. Hence, for example, if particles were important in Brownlee's experiments, $(1/\tau_W) \sim \beta = 2.62\ \text{sec}^{-1}$, whereas if σ represents the significant damping, $(1/\tau_W) \sim \sigma = .0398\ \text{sec}^{-1}$. Although one cannot strictly identify τ_p with τ_W, the dependence of mean chamber pressure on the amplitude of oscillations suggests some connection. Thus, the values of the delay time seem closer to those expected if particle damping were absent. Moreover, later experiments [3] indicated that the delay time was less for longer rockets, a result which must here be related to the reduction of σ with larger L.

Concluding Remarks

It is certainly not a new conclusion that acoustical theory, slightly modified, does accommodate one kind of pressure oscillation observed in rocket engines. But compared to previous work, the analysis discussed here seems to permit a simpler and more orderly assessment of the important factors in a particular case. Interpretation of the data reported by BROWNLEE and MARBLE illustrates the point; the scheme is clearly applicable, more generally, within the approximations cited.

Under other circumstances, different assumptions might be required. For example, it has been proposed that the dissipation of energy associated with propagation of waves in the solid propellant may be significant. Some numerical results have been reported in [24]. Although they do not apply directly to the rockets used by BROWNLEE and MARBLE, they do show a very strong dependence of attenuation on frequency. If such a loss of energy were important, it appears that the stability boundary in Fig. 2 would not have the shape observed. More importantly, it was not necessary to be concerned with erosive burning; apparently at least the weak waves observed by BROWNLEE are caused mainly by the sensitivity of the burning to changes of pressure. If erosion must be considered, as may be the case for strong waves, a different admittance function for the burning surface must be used.

Evidently BROWNLEE's data afford no information concerning the influence of particles suspended in the gas phase. However, it is fairly clear from the present calculations, and well supported by experience reported in the literature, that this source of damping is sufficient to prevent the maintenance and growth of weak waves, and hence also those large amplitude waves which would result. Practically, then, the foregoing analysis is, in a sense, obsolete. On the other hand, there are serious difficulties [4] with other sorts of strong waves which are not, at present, obviously related to acoustic modes. Understanding of those waves must surely be based, at least indirectly, on a thorough knowledge of the simpler acoustics problem.

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